QEA Boat Final Report

Zack Davenport and Colvin Chapman

February 11, 2018

Abstract

This report discusses the process of designing, simulating, and fabricating a model boat that can meet specific performance requirements such as floating flat and having an angle of vanishing stability between 120°and 140°. These requirements are outlined in the *Introduction*, and foundational concepts like the angle of vanishing stability are explained in the *Technical Background* section. To achieve the equation for the surface that defines the boat's hull, we iterated through many equations and plots, which are laid out in the *Boat Hull Design* section. The procedure for mathematically calculating the angle of vanishing stability of the final design involved calculating the boat's center of mass, center of buoyancy, and the righting moment between these two points. This process is further explained in the *Calculating AVS* section. After creating a working simulation in Mathematica for the final hull design, we recreated the boat in SolidWorks and then laser cut and assembled the pieces, as described the *Physical Design and Fabrication* section. When tested, the boat successfully floated flat and was able to right itself for any angle up to 136°. A more detailed analysis of the boat's testing are analyzed in the *Performance and Analysis* section.

1 Introduction

In this project, we designed and mathematically calculated the physical properties of a model boat that could accomplish a predefined set of tasks. The final boat needed to float flat in the water when fully loaded, have an angle of vanishing stability (explained in greater detail in the next section) of between 120°and 140°, have a maximum computed righting moment of at least 0.2 N-m, be as fast as possible, and look as professional as possible. The calculations and overall design conceptualization for this boat were done in Mathematica, and the CAD was done in SolidWorks. The final boat was constructed using laser-cut hardboard, steel weights as ballast, and heat-shrink plastic as an outer surface. Our design was inspired by the camelopard, which is the archaic term for a giraffe, and is therefore aptly named Camelopard the Water Zeppelin.

2 Technical Background

A boat's angle of vanishing stability (AVS) is the angle, θ, at which it can no longer self-right and capsizes. The boat has a moment arm created by the distance between the center of mass (COM) and the center of buoyancy (COB), and the buoyant force acts on this arm to create a torque, as illustrated in Figure 1. When this torque switches from being positive to negative, the boat reaches its AVS and capsizes, as illustrated in Figure 2. When the torque due to buoyancy returns the boat to its original position of stable equilibrium, it is a restoring moment. For this to be the case, the COB must be nearly vertically aligned with the COM. If these two points move

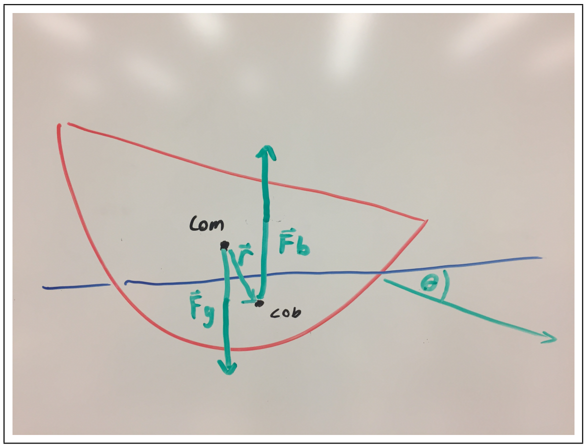

Figure 1: A free-body diagram of boat forces when righting to a stable equilibrium.

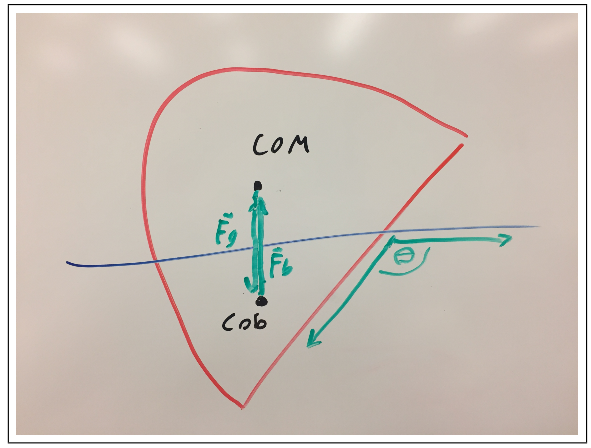

Figure 2: A free-body diagram of boat forces when the boat is at its AVS. Note the alignment of the COM and COB.

slightly out of alignment, a righting torque is created that pushes the two points back to alignment. The next possible angle at which the two points align is an unstable equilibrium, meaning the system will not restore to its original state if slightly offset. When the system is slightly offset from an unstable equilibrium, the torque created will change the angle such that the two points become even farther out of alignment.

3 Boat Hull Design

All the points in the following region define our boat.

$$a \,|\, (\,(q\,(|x|-1\,)\,)^m + 1\,)\,|y|^n + |bx|^p \,| \leq z \leq d \tag{1}$$

This complex three-dimensional inequality was formed by multiple iterations of the Manipulate function in Mathematica (which allowed us to dynamically adjust numerous parameters), each of which included a new term in our inequality with numerical constants instead of parameters. We assumed the positive z dimension would reference the boat's height, x the length, and y the width. Our first-generation power boat was defined by the following inequality. We decided an n value of 2 and a d value of 4 produced the most boat-like cross sections from the visualizations in Mathematica, and therefore continued without the absolute value since it was no longer needed. plotted in Figure 3.

$$|y|^n \leq z \leq d \tag{2}$$

Because this boat had an infinite length, the next step was to define the length as a function of x. We did this by adding the term $+|bx|^p$ to the original y^n term, making the new boat the region defined in Equation 3 below. The parameters p and b are constants that we used to tune the boat curve along the xz-axis to our liking. The absolute value makes the front and back symmetric. Eventually, we settled on a b value of 0.2 and a p, making our boat defined by the region plotted in Figure 4.

$$(y^2 + |bx|)^p \leq z \leq 4 \tag{3}$$

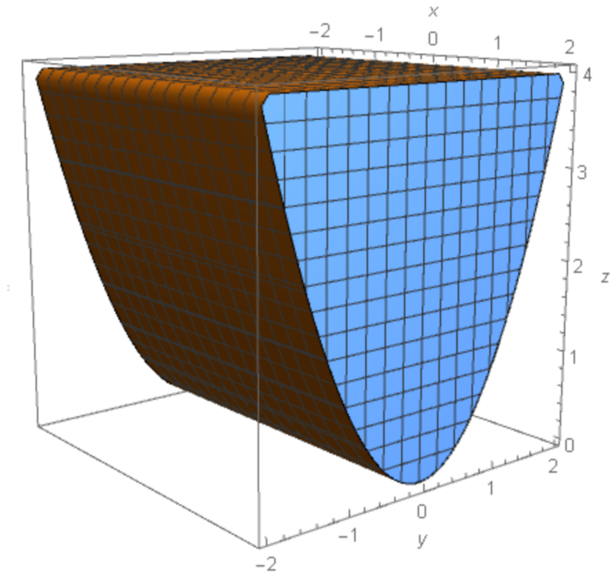

Figure 3: Boat modeled by the extruded parabola derived in Equation 2

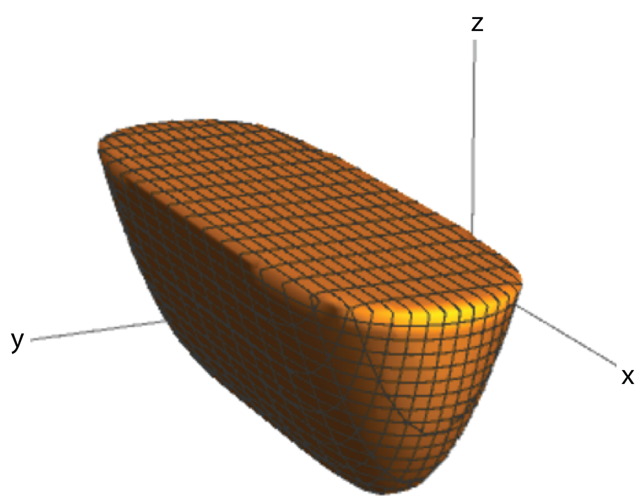

Figure 4: Boat modeled by the extruded parabola with the end term added in Equation 3.

While the region fully defined a somewhat boat-shaped region, we wanted our boat to be more hydrodynamic. We made a temporary parameter, ψ, that would multiply the y^2 term. A higher ψ value makes the boat thinner, but just as long and tall:

$$\psi y^2 + (0.2x)^6 \leq z \leq 4$$

Ideally, this ψ parameter would leave the hull cross-section near the middle about the same as before, but taper the hull to a point at each end, as in Figure 5. This worked best when we defined

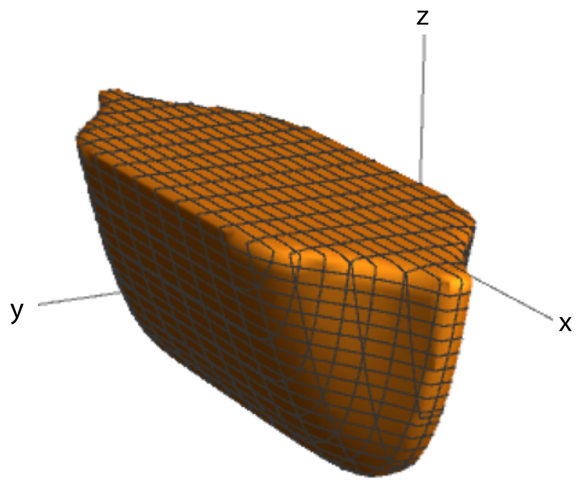

Figure 5: Boat model including tapering ends modeled by Equation 4.

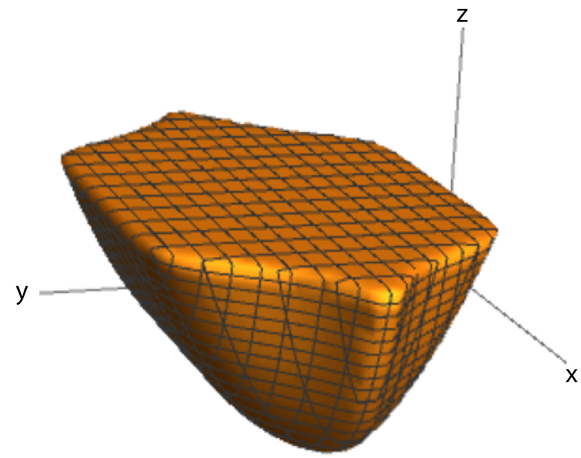

Figure 6: Our final boat shape design - defined from the region in Equations 1 and 6.

ψ as a function of x:

$$\psi = \left(\left(q \left(|x| - 1 \right) \right)^m + 1 \right)$$

The absolute value, as in the previous term, maintains symmetry. The q and the -1 separate the start of the tapering regions. The $+1$ keeps the midsection an approximate extruded parabola (otherwise ψ would be 0 at $x = 1$). For values less than 1 this expression yields complex numbers, so another absolute value was added around the entire first term making the result a real, scalar value. This new entire boat expression is plotted in Figure 5:

$$\left| \left(\left(q \left(|x| - 1 \right) \right)^m + 1 \right) y^2 + (.2x)^6 \right| \leq z \leq 4 \tag{4}$$

After some rough AVS approximations, we concluded that our AVS would be too small to be within the 120°-140° range. To combat this problem, we multiplied the expression in our inequality by the parameter a (between 0 and 1) to widen the cross sections enough to be within the range of a valid AVS.

$$a \left| \left(\left(0.33 \left(|x| - 1 \right) \right)^7 + 1 \right) y^2 + (.2x)^6 \right| \leq z \leq 4 \tag{5}$$

After performing the same rough rough AVS estimations, we decided on an a value of 0.3. This gave the most promising AVS estimates, although a more accurate AVS calculation is explained in the next section that takes other factors into account besides boat shape. The resulting region shown in Figure 6 is the final model of the surface:

$$0.3 \left| \left(\left(0.33 \left(|x| - 1 \right) \right)^7 + 1 \right) y^2 + (.2x)^6 \right| \leq z \leq 4 \tag{6}$$

4 Calculating AVS

Before an AVS can be calculated, components such as the center of mass, the center of buoyancy, and the torque between them for any given angle must be calculated. To find the center of mass of the hardboard components of the boat, m_h, the mass and center of mass for each sub-component must first be known. The mass of our boat comes from three main parts: the steel ballast - 982g - the aluminum mast - 96g - and the hardboard- first estimated to be 326g. This less-accurate estimate does not take into account individual hardboard components like the keel and deck - it only gives calculations that assume solid hardboard cross-sections. More accurate calculations can be made in SolidWorks after a mechanical design is decided.

The total mass of the hardboard can be roughly approximated using $\rho_h = 1.82$ grams per square inch for the density of hardboard, and integrating the area of each horizontal cross section using this density. In Equation 7 below, C represents the equation for the cross-section of the hull as given by Equation 1 at some value x.

$$m_h = \iint_C \rho_h \, dz \, dy \tag{7}$$

Summing the masses of all the components, our initial estimate at our total mass was 1,404g.

Using the same assumptions as the total mass of the hardboard, the center of mass of the hardboard can be approximated using the following equations. Again, SolidWorks will give a more accurate COM after the mechanical design is finished. Because our boat design is symmetric from port to starboard, we assumed the y position of the center of mass to be 0. The same can be true for bow/stern symmetry making the x component of the center of mass 0. Had this not been the case, we would have done a calculation using Equation 10 to find the x-component of the center of mass.

$$COM_y = \frac{1}{m_h} \iint_C \rho_h \, y \, dz \, dy \tag{8}$$

$$COM_z = \frac{1}{m_h} \iint_C \rho_h \, z \, dz \, dy \tag{9}$$

Next, we calculated the center of mass of the entire boat assembly by computing a weighted average based on the relative masses and positions of the components. Our boat is symmetrical in across the XZ plane and the YZ plane (left to right and front to back respectively). This meant that the COM position in the x and y direction was 0. In the z-direction, the center of mass as a weighted average was calculated as follows:

$$\frac{\sum r_i m_i}{\sum m_i} \tag{10}$$

The next step is to define the waterline as a function of depth and angle. The tangent of the inputted θ would be multiplied by y to create the slope of a line, then the intercept defined by the depth would be added. Next, the boat's submerged region can be found by finding the intersecting region between the water and the boat. Now that this region is defined, we can calculate the center of buoyancy of the boat, which can be treated as the center of mass of the submerged region S. This involves calculating the mass of the displaced water, m_w, which is shown below:

$$m_w = \iiint_S \rho_w \, dx \, dy \, dz \tag{11}$$

Next, we calculate the 3-dimensional center of buoyancy (COB) of the boat. This is done by computing the center of buoyancy in both the y-direction (COB_y) and z-direction (COB_z), using β as the equation for the boat hull and using ρ_w as the density of water. The x-direction does not need to be included because the boat has symmetry in this direction and can assumed to be 0. These calculations are as follows:

$$COB_y = \frac{1}{m_w} \iiint_\beta \rho_w y \, dx \, dy \, dz \tag{12}$$

$$COB_z = \frac{1}{m_w} \iiint_\beta \rho_w z \, dx \, dy \, dz \tag{13}$$

Now that both the COM and the COB have been calculated, a moment arm between the two can be made. This moment arm between these two points is again illustrated in Figures 1 and 2. This takes on the form of a vector that takes into account the y and z direction distances from the COM to the COB. This is shown below:

$$\vec{r} = 0\hat{\mathbf{i}} + (COB_y - COM_y)\hat{\mathbf{j}} + (COB_z - COM_z)\hat{k} \tag{14}$$

The force acting on this moment arm ($\vec{r}$) is the buoyant force, which is equal to the weight of the displaced water. Because the boat may be at an angle in the water, this buoyant force must be split up into y and z components using sines and cosines. The vector for the buoyant force is shown below, where g is the force of gravity, $9.8m/s^2$:

$$\vec{F_B} = 0\hat{\mathbf{i}} + (-g \, m_w \, sin\theta)\hat{\mathbf{j}} + (g \, m_w \, cos\theta)\hat{k} \tag{15}$$

To find the AVS, the moment arm vector needs to be crossed with the buoyant force vector to create a torque vector, which can be examined over a range of angles to analyze when the boat loses stability and capsizes. The calculation of the torque vector ($\vec{\tau}$) by cross products is shown below:

$$\vec{\tau} = \vec{r} \times \vec{F_B} \tag{16}$$

Before the plot of torque vs. angle is discussed, it is worth mentioning that the center of mass of the boat can be manually controlled by adding or subtracting from the y- and z- components. By manually adjusting the z-position of the COM, we were able to run a variety of simulations to determine roughly where the COM would ideally need to be to achieve an AVS in the desired range. This ideal COM location affects where in the boat's hull our steel ballasts are placed. Using this information, we can combine the relative centers of masses of the various components into the weighted average equation shown in Equation 10.

When we implemented this procedure, we realized at this point that our hull shape was not wide enough to grant a satisfactory AVS, and so we multiplied our boat hull surface by a factor of 0.3. This moved the center of buoyancy laterally which increased the AVS to a satisfactory range. As seen in Figure 7, the AVS the new hull shape is 132°, after recalculating our center of mass.

5 Physical Design and Fabrication

To create the CAD for our boat, we calculated the equations of the cross-sections from our hull equation in Mathematica and implemented them into SolidWorks using the equation-driven curve feature. This same process was used for the boat's spine and deck, and the pieces were combined into a SolidWorks assembly to determine how they would all fit together, as shown in Figure 8. We then discovered it was necessary to cut out the upper portions of the cross-sections, effectively lowering the center of mass of the assembly. We tested this by using the center of mass

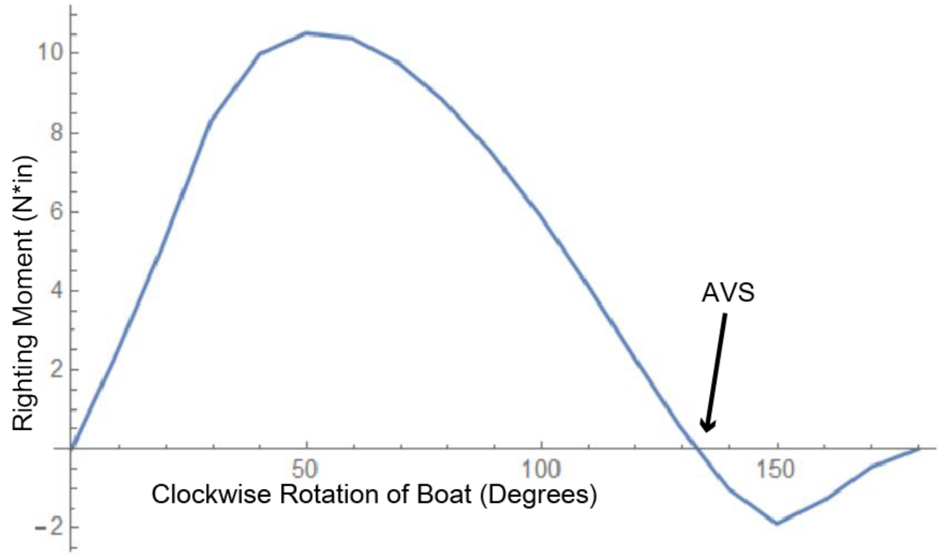

Figure 7: The AVS curve plots the torque vs. rotation angle of the boat to determine when its torque becomes negative and capsizes. Based on this plot, the AVS of the boat was determined to be around 132°.

estimation feature in SolidWorks and then implemented this into our weighted COM calculation in Mathematica (Equation 10). This allowed us to verify that our overall placement of the ballasts would function as we had intended and give us the desired AVS.

The boat pieces were put into a drawing and laser cut on a 1/8" sheet of hardboard. We first tested the pieces for fit, roughly attached them using hot glue, and then solidified the connections using wood glue. Our design required our masses to be placed between the cross-sections while the boat was being assembled, which allowed them to simultaneously be glued securely in place.

After the adhesive set for a day, we began the process of covering the hull with heat-shrink plastic. First, a blue sheet was cut out to fit the size of the deck and ironed on. A larger yellow sheet to cover the hull was cut to size, ironed on, and then heat-shrunk to the hull using a craft iron and a heat gun. The giraffe figurehead, blue spots, and a vinyl-cut decal were later added for aesthetic purposes.

6 Performance and Analysis

On testing day, our boat was placed in the pool and floated perfectly level with the surface of the water, as shown in Figure 10. This indicated that the masses we used were well-centered in the boat. Because our boat had the smallest hull, we had less of a range as to where we could vertically place our masses, and we likely had a greater submerged depth. Despite these potential challenges, our boat successfully met all the criteria when tested. Based on our calculations and simulations, we had predicted an AVS of around 132°. After testing, it was determined that the AVS was actually around 136°. We could think of a few reasons for this slight discrepancy: when measuring the diameters of our steel ballasts, we slightly overmeasured, which created a small gap around the masses in the assembly. Our original calculations, therefore, had assumed the masses to be placed slightly higher, which would have raised the predicted COM and likely lowered the predicted AVS.

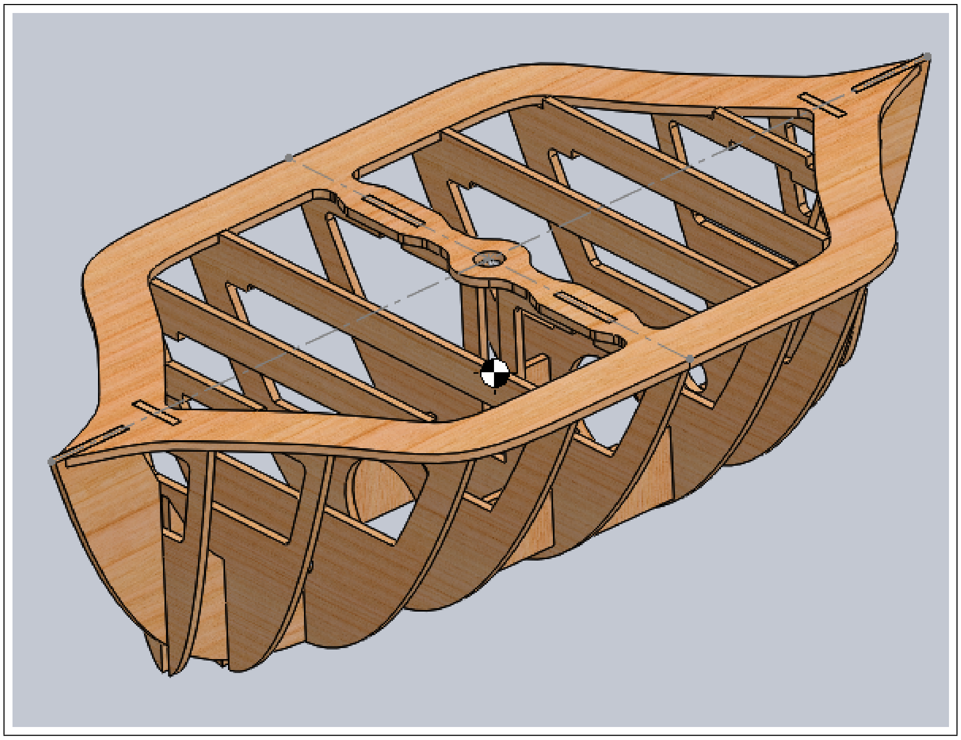

Figure 8: The completed SolidWorks allowed us to calculate the COM for all of the hardboard pieces. The COM is represented in the assembly by the black and white dot.

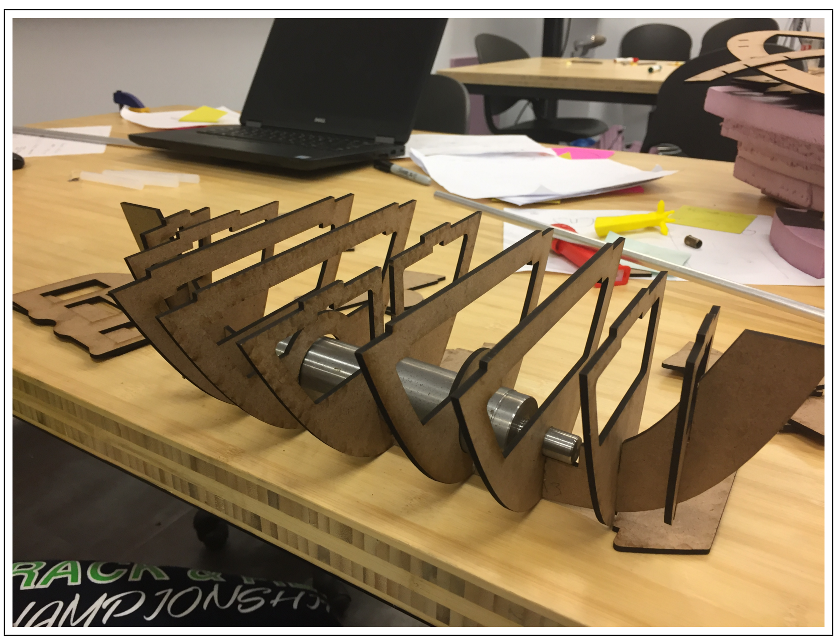

Figure 9: The hardboard assembly is being constructed with the steel ballasts in place.

We may have also used more hot glue and wood glue than we had taken into account, which likely lowered our actual COM and increased the observed AVS.

Other construction elements that could have had a similar effect on the COM and AVS include the camelopard figurehead and the heat-shrink plastic casing on the boat. When our boat was tested for speed, it reached its maximum velocity at 0.468 m/s. At the time of testing, this was the top speed and was ultimately the third-fastest boat. We attribute our high speed to the pointy shape of our hull which was very effective at pushing water out of the way to reduce drag.

This project allowed for the combination of creative mathematical design with exploration into some of the fundamental topics of mechanics. By effectively learning to use computer resources such as Mathematica and SolidWorks and by exploring fabrication techniques like laser cutting and heat-shrinking, we were able to successfully produce a boat that both met the performance requirements and was an aesthetically pleasing representation of the glorious camelopard.

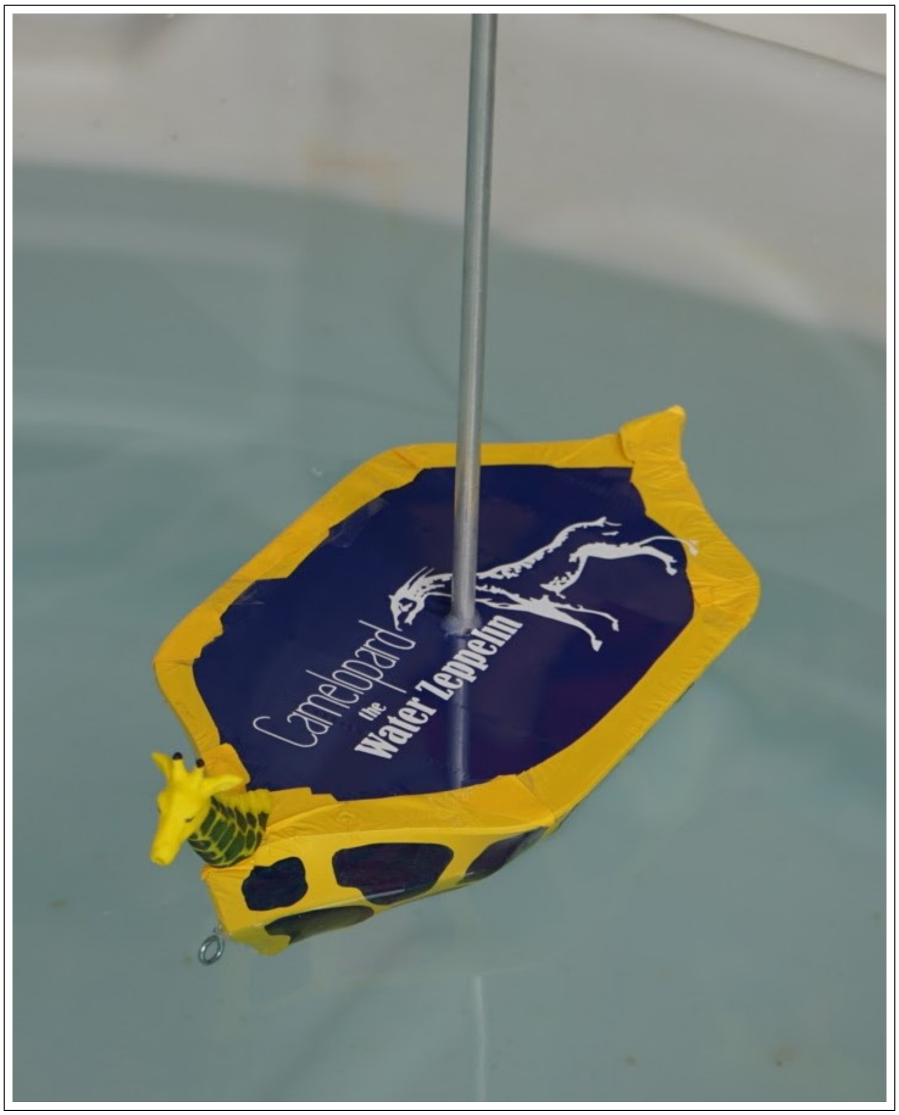

Figure 10: Camelopard the Water Zeppelin, in her completed form, is tested in the pool to see if she floats level with the water.